ON THE RELATIONS OF ATMOSPHERIC PRESSURE, TEMPERATURE, AND DENSITY TO ALTITUDE.

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[Dated: Weather Bureau, Washington, Apr. 22, 1919.]

The relations of atmospheric pressure, temperature, and density to altitude are of importance to aviators and to aeronautical engineers, in order that laboratory tests may be correlated with performance under flight conditions. As a basis for these relations, the meteorologist has to offer the average free air atmospheric temperature, pressure, and humidity as determined by means of recording instruments carried to the desired altitudes by kites or balloons, and the hypsometric formula, which latter represents with great accuracy the variation of pressure with height when we know the temperature and humidity variations.

Unfortunately, observations show that the relations of temperature and humidity to height vary from day to day, and from hour to hour of the same day, especially below an altitude of about 3,000 meters (10,000 feet), so that the actual relations at any time rarely or never correspond to the average conditions. For this reason engineers generally prefer to employ simpler expressions for the relations of air temperature and pressure to height than are given by observations and the hypsometric formula, respectively. For the temperature-altitude relation they generally prefer a straight-line relation, as, for example, a fall of 1° F. per 300 feet increase in elevation; and in the equation expressing the pressure-altitude relation the humidity term is often omitted.

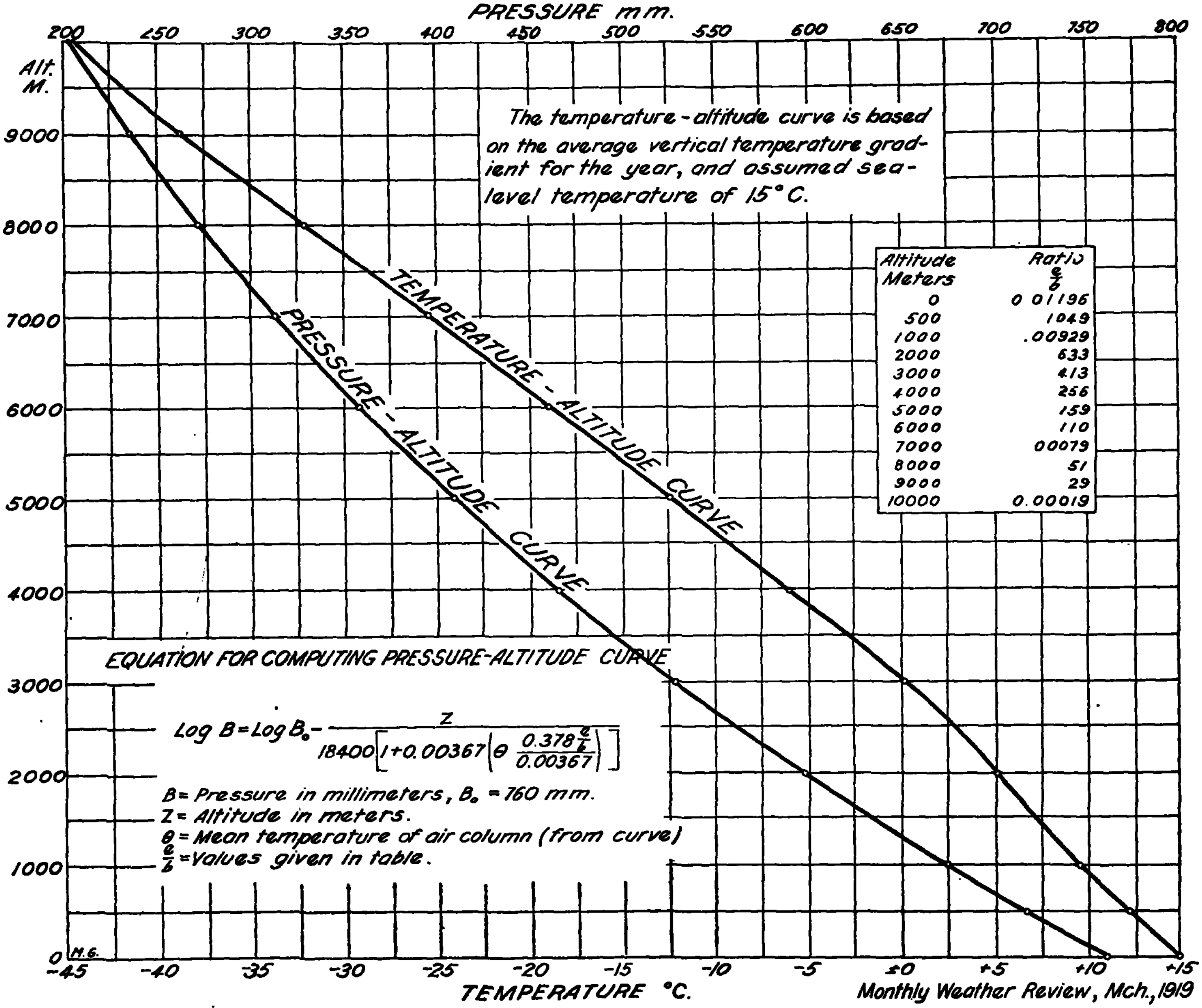

FIG. 1.—Temperature-altitude and pressure-altitude curves. (A plus sign should follow θ in the equation above.)

It is possible to select a straight-line temperature-altitude relation that will approximate actual relations up to a height of about 10,000 meters (30,000 to 40,000 feet). Above this height, however, as the lower limit of the stratosphere is approached, the decrease in temperature with increase in altitude becomes rapidly less, so that a straight-line relation that applies in the lower layers of the troposphere will here give temperatures too low by an amount that increases rapidly with altitude. Therefore, it must be understood that these simplified equations can be employed to express atmospheric temperature, pressure, and density relations to height, only up to the altitude given above.

Table 1, in metric measures, and Table 2, in English measures, give atmospheric temperatures and densities at different heights under approximately average conditions for the east-central part of the United States. The assumed sea-level pressure, B_0, temperature, t_0, and vapor pressure e_0, are, respectively, $B_0 = 760$ mm. (29.92 in.), $t_0 = 15°$ C. (59° F.), and $e_0 = 9.1$ mm. (0.36 in.), and the assumed vertical distribution of temperature and water vapor corresponds to the annual means given by Gregg,[1] and reproduced in figure 1. This vertical distribution of temperature is also in close agreement with that given by Humphreys [2] for the warm part of the year. The hypsometric formula employed in computing the air pressure, B, is given in the figure, and in the heading of each table.

For θ, the mean temperature of the air column obtained by means of the equation that follows has been employed:

$$\theta = \frac{t_0 + 2t_1 + 2t_2 + \ldots\ldots + 2t_{n-1} + t_n}{2n}$$

where $t_0, t_1, t_2, \ldots\ldots t_{n-1}, t_n$, represent the temperatures from the bottom to the top of the air column observed at equal intervals of height, as 500 meters or 1,000 feet. Actually, the temperature should be obtained by taking the harmonic mean of the temperature with respect to altitude, thus:

$$\frac{2n}{T} = \frac{1}{T_0} + \frac{2}{T_1} + \frac{2}{T_2} + \ldots\ldots + \frac{2}{T_{n-1}} + \frac{1}{T_n}$$

where $T = \theta + 273°$ C., or $\theta + 459.4°$ F., and $T_0 = t_0$, $T_1 = t_1$, etc. expressed in degrees on the Absolute scale. Unless the temperatures cover a wide range, however, the increased accuracy of the result obtained from the use of the harmonic mean in place of the arithmetical mean does not compensate for the additional labor involved, and especially when we are dealing with approximate mean values only.

The equation for computing the atmospheric density, ρ, which is also given in the headings of Tables 1 and 2, gives for the average surface density 1.220 kilograms per cubic meter, as compared with 1.293 kilograms per cubic meter for dry air at tempearture 0° C. under 760 millimeters pressure, which is universally accepted as *standard* air density.

[1] Gregg, Willis R. Mean values of free-air barometric and vapor pressures, temperatures, and densities over the United States. MONTHLY WEATHER REVIEW, January, 1918, 46:11-21.

[2] Humphreys, W. J. Temperatures, pressures, and densities of the atmosphere at various levels in the region of northeastern France. This REVIEW, p. 159, fig. 1.

TABLE 1.—*Average air temperatures and densities at different altitudes.*[1] *Metric measures.*

$$\text{Log. } B = \text{Log. } B_0 - \frac{Z}{18400\left[1 + 0.00367\left(\theta + \dfrac{0.378\frac{e}{b}}{0.00367}\right)\right]}$$

ρ = density = $\frac{B - 0.378e}{T} \times 0.35921$; standard density = 1.293 kg./cu. m.

Altitude.	Pressure.	Temperature.	Vapor pressure.	Atmospheric density.	
				Per cent standard.	Per cent surface.
Meters.	*mm.*	°*C.*	*mm.*		
0	760.0	15.0	9.10	94.4	100.0
500	716.2	12.3	7.54	89.8	95.2
1,000	674.5	9.6	6.29	85.4	90.5
2,000	597.4	5.1	3.80	77.0	81.6
3,000	527.9	0.1	2.19	69.3	73.5
4,000	465.3	− 6.0	1.20	62.5	66.3
5,000	408.9	−12.5	0.65	56.4	59.7
6,000	358.2	−19.0	0.40	50.6	53.7
7,000	312.9	−25.4	0.25	45.4	48.1
8,000	272.3	−32.1	0.14	40.6	43.0
9,000	236.2	−38.7	0.07	36.2	38.4
10,000	204.2	−44.4	0.04	32.1	34.0

[1] Computed for sea-level pressure 760 millimeters, and sea-level temperature 15°C.

TABLE 2.—*Average air temperatures and densities at different altitudes.*[1] *English measures.*

$$\text{Log. } B = \text{log. } B_0 - \frac{Z}{60367\left[1 + 0.002039\left(\theta - 32 + \dfrac{0.378\frac{e}{b}}{0.002039}\right)\right]}$$

ρ = density = $\frac{B - 0.378e}{T}$ 16.124; $T = 459.4 + t$; standard density = 1.293 kg./cu. m.

Altitude.	Pressure.	Temperature.	Vapor pressure.	Atmospheric density.	
				Per cent standard.	Per cent surface.
Feet.	*Inches.*	°*F.*	*Inches.*		
0	29.92	59.0	0.36	94.4	100.0
1,000	28.86	56.0	.32	91.6	97.1
2,000	27.83	52.9	.28	88.9	94.2
3,000	26.83	49.8	.25	86.2	91.4
4,000	25.86	47.3	.22	83.6	88.6
5,000	24.92	44.9	.19	80.9	85.8
6,000	24.02	42.5	.16	78.4	83.1
7,000	23.12	40.0	.14	75.9	80.4
8,000	22.28	37.3	.12	73.5	77.9
9,000	21.46	34.3	.105	71.3	75.5
10,000	20.66	31.2	.086	69.1	73.2
11,000	19.88	27.9	.072	66.9	70.9
12,000	19.13	24.6	.060	64.8	68.7
13,000	18.40	21.3	.050	62.8	66.6
14,000	17.69	17.8	.045	60.8	64.5
15,000	17.00	14.2	.040	58.9	62.4
16,000	16.35	10.8	.035	57.1	60.5
17,000	15.72	7.4	.030	55.3	58.6
18,000	15.10	4.0	.025	53.5	56.7
19,000	14.50	0.4	.020	51.8	54.9
20,000	13.92	− 3.4	.017	50.1	53.1
21,000	13.36	− 7.2	.015	48.5	51.4
22,000	12.82	−11.0	.013	47.0	49.8
23,000	12.29	−14.8	.010	45.4	48.1
24,000	11.79	−18.4	.008	43.9	46.5
25,000	11.30	−22.0	.007	42.4	45.0
26,000	10.83	−25.6	.006	41.0	43.5
27,000	10.37	−29.2	.005	39.6	42.0
28,000	9.93	−32.6	.004	38.2	40.5
29,000	9.51	−36.0	.003	36.9	39.1
30,000	9.10	−39.2	.002	35.4	37.7

[1] Computed for sea-level pressure 29.92 inches, and sea-level temperature 59 °F. (15°C.).

It is of interest to compare the data of Tables 1 and 2 with those obtained by use of the simpler equations sometimes employed.

If we assume the same sea-level conditions as above, but disregard the term containing e in computing the density, we obtain $\rho = 1.226$ kg. per cubic meter, or about 0.5 per cent too high.

Again, if we assume the vertical temperature gradient to be 1° F. per 300 feet, we obtain for an altitude of 30,000 feet, $t = 59° - 100° = -41°$ F., which is only 1.8° F. lower than the value given in Table 2. At 10,000 feet, however, we obtain $t = 59° - 33.3° = 25.7°$ F., which is 5.5° F. lower than the value given in Table 2.

Johnson [3] has shown that the following assumed simple relations of air pressure, temperature, and density to height, give reasonably accurate results.

From the well-known Charles's law we have

$$P\frac{v}{t} = Pv^y = R = \text{constant};$$

$$v = \left(\frac{R}{P}\right)^{1/y};$$

$$W = 1/v = \left(\frac{P}{R}\right)^{1/y} = \text{weight per unit volume.}$$

Let $R^{-1/y} = c$.

Then $W = cP^{1/y}$ .... (1)

from which c may be computed [4].

We may also write

$$-dP = cP^{1/y}dh$$

or

$$-P^{-1/y} = c\frac{dh}{dP}$$

Integrating, $-\dfrac{y}{y-1}P^{\frac{y-1}{y}} = ch + \text{constant}$,

and $P^{\frac{y-1}{y}} = \text{constant} - \dfrac{y-1}{y}ch$.

When $h = 0$,

$$P^{\frac{y-1}{y}} = P_0^{\frac{y-1}{y}}.$$

Therefore $P^{\frac{y-1}{y}} = P_0^{\frac{y-1}{y}} - \dfrac{y-1}{y}ch$ .... (2)

Also, $RT = Pv = \dfrac{P^{\frac{y-1}{y}}}{c}$, and $T_0 = \dfrac{P_0^{\frac{y-1}{y}}}{cR}$

Combining this last equation with (2), we obtain

$$T = \frac{P_0^{\frac{y-1}{y}} - \frac{y-1}{y}ch}{cR} = T_0 - \frac{\frac{y-1}{y}h}{R} \quad \text{.................. (3)}^5$$

[3] Johnson, A. Atmospheric conditions affecting power. Aerial Age Weekly, Mar. 31, 1919, 9:166.

[4] Let $P_0 = 2116.2$ pounds per square foot, $t_0 = 59°$ F., and $y = 1.20$. Then $W_0 = 0.07651$ pounds per cubic foot, and $c = W_0 P_0^{-1/1.2} = 0.00012955$.

[5] $R = \dfrac{P_0}{W_0 T_0} = \dfrac{2116.2}{0.07651 \times 518.4} = 53.354.$

Assuming $y = 1.20$

Equation (1) becomes $W = cP^{1/1.2}$

Equation (2) becomes $P^{1/6} = P_0^{1/6} - \dfrac{ch}{6}$

$= P_0^{1/6} - 0.0000216h$ (P_0 in pounds per cubic foot, h in feet).

For temperatures, t, in Fahrenheit degrees, equation (3) becomes $t = t_0 - \dfrac{h}{6R} = t_0 - 0.003124h$ (h in feet).

TABLE 3.—*Average air temperatures and densities at different altitudes. English mea[illegible].*

$$P^{1/6} = P_0^{1/6} - \frac{ch}{6}; \qquad t = t_0 - \frac{h}{6R}$$

Height.	Pressure.		Density.	Temperature.	Departure from mean observed temperature.
Feet.	*Inches of mercury.*	*Pounds per cubic foot.*	*Per cent.*	°*F.*	°*F*
0	29.92	2,116	100.0	59.0	±0.0
1,000	29.85	2,041	97.0	55.9	−0.1
2,000	27.82	1,968	94.1	52.8	−0.1
3,000	26.82	1,897	91.3	49.6	−0.2
4,000	25.85	1,828	88.5	46.5	−0.8
5,000	24.90	1,761	85.8	43.4	−1.5
6,000	23.99	1,697	83.2	40.3	−2.2
7,000	23.10	1,634	80.6	37.1	−2.9
8,000	22.24	1,573	78.1	34.0	−3.3
9,000	21.41	1,514	75.6	30.9	−3.4
10,000	20.60	1,457	73.3	27.8	−3.4
11,000	19.82	1,402	71.0	24.6	−3.3
12,000	19.07	1,349	68.7	21.5	−3.1
13,000	18.34	1,297	66.5	18.4	−2.9
14,000	17.63	1,247	64.3	15.3	−2.5
15,000	16.94	1,198	62.2	12.1	−2.1
16,000	16.28	1,151	60.2	9.0	−1.8
17,000	15.64	1,106	58.2	5.9	−1.5
18,000	15.02	1,062	56.3	+ 2.8	−1.2
19,000	14.42	1,020	54.4	− 0.4	−0.8
20,000	13.84	979	52.6	− 3.5	−0.1
21,000	13.28	939	50.8	− 6.6	+0.6
22,000	12.74	901	49.1	− 9.7	+1.3
23,000	12.22	864	47.4	−12.9	+1.9
24,000	11.72	829	45.8	−16.0	+2.4
25,000	11.23	794	44.2	−19.1	+2.9
26,000	10.76	761	42.6	−22.2	+3.4
27,000	10.31	729	41.1	−25.3	+3.9
28,000	9.87	698	39.7	−28.5	+4.1
29,000	9.45	668	38.3	−31.6	+4.4
30,000	9.04	639	36.9	−34.7	+4.5
36,960	....	....	....	−56.5	+2.7
40,000	....	....	....	−66.0	−6.2

Table 3 gives the pressure in inches of mercury and pounds per square foot, (the latter computed from equation (2), and then converted into inches of mercury); the temperature in degrees F, computed from equation (3); and the resulting density, in percentages of surface density.

Comparison with Table 2 shows a maximum difference in pressures of 0.6 per cent at a height of 22,000 feet, and a difference of 0.9 per cent in the densities at altitudes of 26,000 and 27,000 feet. The temperatures are too low up to 20,000 feet, with a maximum difference of 3.4° at 9,000 and 10,000 feet, and are too high between about 20,000 and 38,000 feet. Above this latter elevation equation (3) does not apply.

A slight change in y, and a resulting change in c, will adapt equations (2) and (3) to any desired vertical temperature gradient. Thus, $y = 1.19$ corresponds very closely to a gradient of 3°F. per 1,000 feet; and $y = 1.21$, very closely to a gradient of 1°F. per 300 feet.